Self-similar solutions for dam breaks computed by *SWASHES*.

François JAMES, Carine LUCAS,
swashes.contact@listes.univ-orleans.fr

2014-02-20

In this document, we detail the self-similar solutions for viscous shallow water equations with a laminar friction law proposed in [1] (based on [2]) and programmed in *SWASHES*.

1 Self-similar solution on a flat bottom

This solution is obtained in [1, section 5.2]. The idea is to consider that, at the initial time, the fluid is confined in a domain centered in $x = 0$, with a fixed height, and to let the fluid evolve. Except for small times, the fluid will have a parabolic shape that fits well with the self-similar solution. Note that, as then the problem is symmetric with respect to the y-axis, one can only consider positive x, and the complete solution will be given by the symmetrical function.

The Shallow-Water equations read[1]:

$$\begin{cases} \partial_t h + \partial_x q = 0, \\ \partial_t q + \partial_x \left(\dfrac{q^2}{h} + \dfrac{g}{2} h^2 \right) = -3\nu \dfrac{q}{h^2}, \end{cases} \tag{1}$$

where h is the water height, q the discharge, g the gravity and 3ν the laminar friction coefficient, where ν is the kinematic viscosity. If we perform the diffusive wave approximation, that is we balance the pressure and friction terms neglecting the inertial terms, we get:

$$\partial_t h + k \partial_x \left(h^3 \partial_x h \right) = 0, \text{ with } k = -\frac{g}{3\nu}.$$

The self-similar solution $\hat{h}$ is a rescaled solution. More precisely, if $h = H\hat{h}$, $x = \frac{1}{H}\hat{x}$ and $t = \frac{1}{H^5}\hat{t}$, denoting by $\eta = \frac{\hat{x}}{\hat{t}^{1/5}}$ the similar variable, the self-similar solution is given by

$$\begin{cases} \hat{h}(\eta, \hat{t}) = \dfrac{1}{\hat{t}^{1/5}} \left(-\dfrac{3}{5k} \left(C_1 - \frac{1}{2}\eta^2 \right) \right)^{1/3} \text{ for } \eta \in [0, \sqrt{2C_1}], \\ \hat{h}(\eta, \hat{t}) = 0 \text{ else.} \end{cases}$$

The constant C_1 is obtained thanks to the mass conservation, that is:

$$2 \int_0^{\sqrt{2C_1}} \left(-\frac{3}{5k} \left(C_1 - \frac{1}{2}\eta^2 \right) \right)^{1/3} d\eta = \text{mass of the fluid.}$$

In *SWASHES*, we have chosen the following values :

- to be consistent with the other solutions, the domain is translated from [-10, 10] to [0, 20],
- the final time is $T = 30$ s,

[1] note that, with the laminar friction law, the momentum equation should read :

$$\partial_t q + \partial_x \left(\frac{6}{5} \frac{q^2}{h} + \frac{g}{2} h^2 \right) = -3\nu \frac{q}{h^2}.$$

- $h(t=0) = 0.4 \times \mathbb{1}_{[7.5,12.5]}$ and the mass of the fluid is equal to 2,
- the kinematic viscosity is $\nu = 0.1$,
- then C_1 is solution of $\int_0^{\sqrt{2C_1}} \left(-\frac{3}{5k}\left(C_1 - \frac{1}{2}\eta^2\right)\right)^{1/3} d\eta = 1$. The value of C_1 can be computed using `sagemath` for example:

```
reset()
var('c','k')
assume (c>0), assume (k>0) # warning: k = Re/(3 Fr^2)
s=solve(integrate(3/(5*k)^(1/3)*(c-1/2*x^2)^(1/3), x, 0, sqrt(2*c))==1,c)
for sol in s :
    if imag_part(sol.rhs())==0 :
        print sol
        res = sol.rhs()
```

and we get: $C_1 = \left(\dfrac{\sqrt{2}}{\text{beta}\left(\frac{1}{2},\frac{4}{3}\right)}\right)^{6/5} \left(-\frac{5k}{3}\right)^{2/5} \approx 0.811774 \left(-\frac{5k}{3}\right)^{2/5}$.

2 Self-similar solution on an inclined plane

We recall here the solution given in [2] and [1, section 5.1]. At the initial time, the fluid has a constant height in a domain $[0, L]$, over an inclined plane given by $z_b = \alpha x + \beta$ with $\alpha < 0$. Under these conditions, the Shallow-Water equations read:

$$\begin{cases} \partial_t h + \partial_x q = 0, \\ \partial_t q + \partial_x \left(\dfrac{q^2}{h} + \dfrac{g}{2}h^2\right) = -gh\partial_x z_b - 3\nu \dfrac{q}{h^2}, \end{cases} \tag{2}$$

where h is the water height, q the discharge, g the gravity and 3ν the laminar friction coefficient, where ν is the kinematic viscosity. If we perform the kinematic wave approximation, *i.e.* we neglect the inertial and pressure terms in the momentum balance, we obtain:

$$\partial_t h + k\partial_x \left(\frac{h^3}{3}\right) = 0, \text{ with } k = -\frac{\alpha g}{\nu} > 0.$$

The self-similar solution can be obtained thanks to the method of characteristics. We get that h is constant along the characteristics given by $\dfrac{dx}{dt} = kh^2$, and $x = x_0 + kh^2_{|t=0,x0}t$, x_0 being the initial value of the characteristic. Then, the solution h is

$$h = \sqrt{\frac{x - x_0}{kt}} \underset{x \gg x_0}{\longrightarrow} \sqrt{\frac{x}{kt}}. \tag{3}$$

The mass stay unchanged during the movement, such that, if A denotes the initial mass

$$A = \int_0^{+\infty} h(t=0,x)\,dx,$$

the fluid profile ends abruptly at $x_F = \left(\dfrac{9A^2kt}{4}\right)^{1/3}$ and $h_F = h(x_F, t) = \dfrac{1.5A}{x_F}$. The profile could be smoothed off including the effects of the surface tension around x_F (this regularization is not implemented in SWASHES).

Remark 1 *Note that a more general formula can be obtained rescaling the equations, without any assumption on the localization of the fluid at the initial time. In that case, if* $h = H\hat{h}$, $x = \frac{1}{H}\hat{x}$ *and*

$t = \frac{1}{H^3}\hat{t}$, denoting by $\eta = \frac{\hat{x}}{\hat{t}^{1/3}}$ the similar variable, the self-similar solution is given by

$$\hat{h}(\eta,\hat{t}) = \hat{t}^{-1/3}\frac{\sqrt[3]{2}\left(9H_0^3k^3+\sqrt{81H_0^6k^6-12\eta^3k^3}\right)^{2/3}+2\sqrt[3]{3}\eta k}{k\sqrt[3]{6^2}\left(9H_0^3k^3+\sqrt{81H_0^6k^6-12\eta^3k^3}\right)^{1/3}},$$

where H_0 is such that $\hat{h}(0,\hat{t}) = H_0\hat{t}^{-1/3}$. Here, $H_0 = 0$.

In *SWASHES*, we have chosen the following values :

- to be consistent with the other solutions, the domain is translated from [-2, 18] to [0, 20],
- the final time is $T = 100$ s,
- $h(t=0) = 0.1 \times \mathbb{1}_{[2,12]}$ and the mass of the fluid A is equal to 1,
- the kinematic viscosity is $\nu = 0.1$,
- the topography is $z_b = -0.1x + 3$.

References

[1] Serge Hèzièwè Bodjona, Etude d'exemples simples d'écoulements, comparaison Saint-Venant et Navier-Stokes. *Master's thesis*, Mécanique des Fluides, Fondements et Applications, Université P. & M. Curie, France, 2013.

[2] Herbert E. Huppert. Flow and instability of a viscous current down a slope. *Nature*, 300:427–429, 1982. doi:10.1038/300427a0.