Validation on swash analytic solutions.

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We present here the solutions obtained with *FullSWOF_1D* for the test cases of Carrier and Greenspan [1], thanks to the work of Gaveau [2].

1 Transient solution: damping of the wave on a beach

In this section, we are interested in a damping wave on a beach, defined by $z = \alpha x$ for $x \in [0, L]$. The analytic solution is then given, at time t_f and at the point x by (see [1]):

$$\begin{cases} \tilde{\lambda} = \dfrac{2}{a}\left(v(\tilde{\sigma},\tilde{\lambda}) + t_f\right), \\ \dfrac{x}{L} = -\dfrac{\tilde{\sigma}^2 a^2}{16} + \eta(\tilde{\sigma},\tilde{\lambda}) + x_0, \\ \eta(\tilde{\sigma},\tilde{\lambda}) = -\dfrac{v^2}{2} + e\mathcal{R}e\left(1 - \dfrac{5-4i\tilde{\lambda}}{2\left((1-i\tilde{\lambda})^2+\tilde{\sigma}^2\right)^{3/2}} + \dfrac{3}{2}\dfrac{(1-i\tilde{\lambda})^2}{\left((1-i\tilde{\lambda})^2+\tilde{\sigma}^2\right)^{5/2}}\right), \\ v(\tilde{\sigma},\tilde{\lambda}) = \dfrac{8e}{a}\mathcal{I}m\left(\dfrac{1}{\left((1-i\tilde{\lambda})^2+\tilde{\sigma}^2\right)^{3/2}} - \dfrac{3}{4}\dfrac{1-i\tilde{\lambda}}{\left((1-i\tilde{\lambda})^2+\tilde{\sigma}^2\right)^{5/2}}\right), \\ h = \left(\eta + x_0 - \dfrac{x}{L}\right)\alpha L, \end{cases}$$

with L is the length of the domain, α the slope of the beach, $a = \frac{3}{2}(1+0.9e)^{1/2}$ where e characterize the initial curvature of the water profile, and x_0 is a shift on the abscissas.
In this system, η is the elevation of the free surface, v the velocity of the fluid, and h the water height.

The initial water height is the analytic solution for $v = 0$ m/s and $t_f = 0$ s (*i.e.* $\tilde{\lambda} = 0$), see Figure 1. It reads:

$$\begin{cases} \eta(\tilde{\sigma},0) = e\left(1 - \dfrac{5}{2}\dfrac{1}{(1+\tilde{\sigma}^2)^{3/2}} + \dfrac{3}{2}\dfrac{1}{(a^2+\tilde{\sigma}^2)^{5/2}}\right), \\ \dfrac{x}{L} = -\dfrac{\tilde{\sigma}^2 a^2}{16} + \eta(\tilde{\sigma},0) + x_0, \\ h = \left(\eta + x_0 - \dfrac{x}{L}\right)\alpha L. \end{cases}$$

The flow evolves to a horizontal steady state, see Figure 2. The boundary condition on the right is not seen (due to the beach) and can be chosen as a wall; on the left boundary condition, we impose the water height.

In *SWASHES*, the length of the domain is set to $L = 20$ m, the slope of the beach is $\alpha = 0.02$, $x_0 = 0.7$ m (to avoid negative abscissas), the initial curvature is characterized by $e = 0.1$, see Figure 1.

The final time is fixed equal to $t_f = 15$ s. In Figure 3, we plotted both the analytic solution and the water height and discharge obtained by *FullSWOF_1D* with 100 points: one can note the very good agreement between the two curves for the water height, a little difference for the discharge that disappears when we refine the mesh.

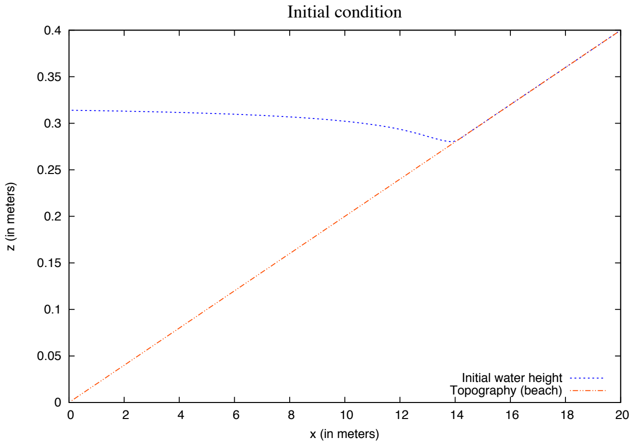

Figure 1: Initial condition (water height).

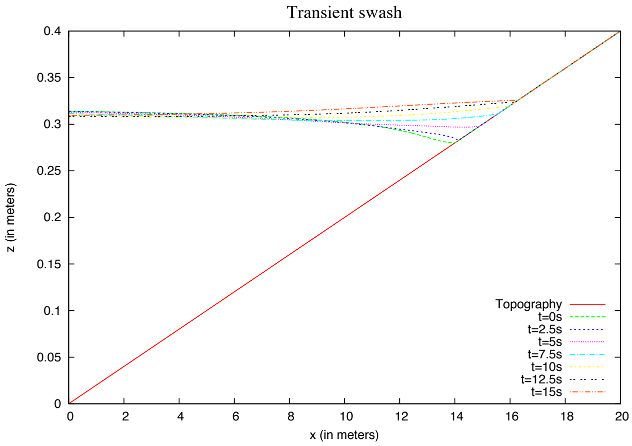

Figure 2: Evolution of the water surface.

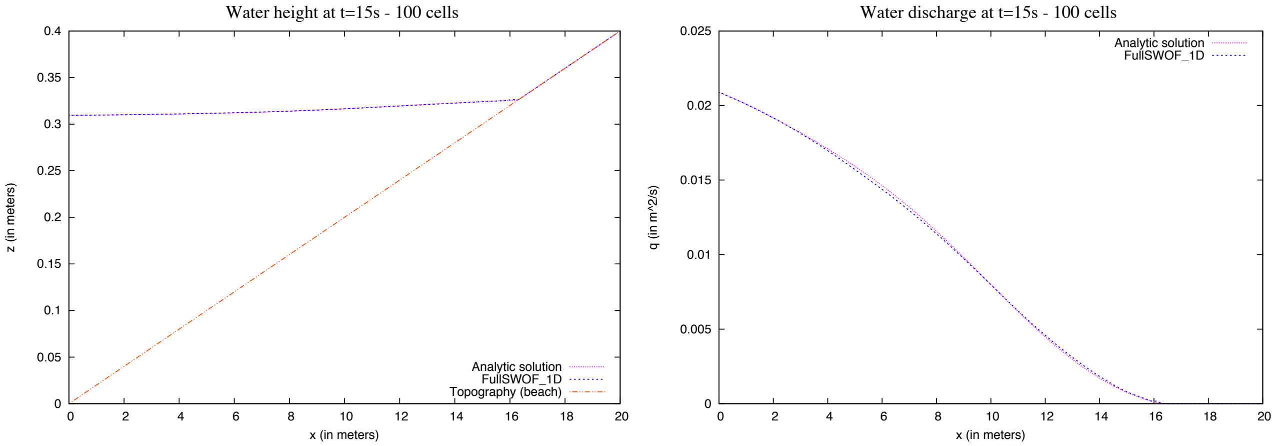

Figure 3: Results at $t_f = 15$ s.

Remark 1 (Imposed height on the left boundary)**.** *The boundary condition "Imposed height" has been modified locally in the code* FullSWOF_1D *to fit this example: it has to be a variable (in time) condition. The values are taken as a piecewise constant function (modified each 0.01s, that is the time step used for Figure 3), given by the corresponding analytic solution. More precisely, we defined the left boundary condition from the file transient_leftbc.txt given by* SWASHES *with nxcell=10000, namely:*

```
if (time<0.01){ hfix = 0.314025; qfix =1.39079e-05; }
else if (time<0.02){ hfix = 0.314025; qfix =2.78157e-05; }
else if (time<0.03){ hfix = 0.314025; qfix =4.17237e-05; }
...
else if (time<14.98){ hfix = 0.309463; qfix =0.020896; }
else if (time<14.99){ hfix = 0.309471; qfix =0.02089; }
else { hfix = 0.309478; qfix =0.020884; }
```

2 Periodic solution

We also studied an example of periodic case on the beach, still defined by $z = \alpha x$ for $x \in [0, L]$. In that case, for a frequency equal to 1, the analytic solution is given, at time t_f and at the point x by (see [1]):

$$\begin{cases} \lambda = 2\left(v(\sigma,\lambda) + t_f\right), \\ \dfrac{x}{L} = -\dfrac{\sigma^2}{16} + \eta(\sigma,\lambda) + x_0, \\ \eta(\sigma,\lambda) = \dfrac{A}{4}\mathcal{J}_0(\sigma)\cos(\lambda) - \dfrac{v^2}{2}, \\ v(\sigma,\lambda) = -\dfrac{A}{\sigma}\mathcal{J}_1(\sigma)\sin(\lambda), \\ h = \left(\eta + x_0 - \dfrac{x}{L}\right)\alpha L, \end{cases}$$

where we used the same notations as before, *i.e.* L is the length of the domain, α the slope of the beach, $a = \frac{3}{2}\left(1 + 0.9e\right)^{1/2}$ where e characterize the initial curvature of the water profile, x_0 is a shift on the abscissas, η is the elevation of the free surface, v the velocity of the fluid, and h the water height. The constant A is the amplitude of the solution and $\mathcal{J}_0$ and $\mathcal{J}_1$ are the first two Bessel functions.

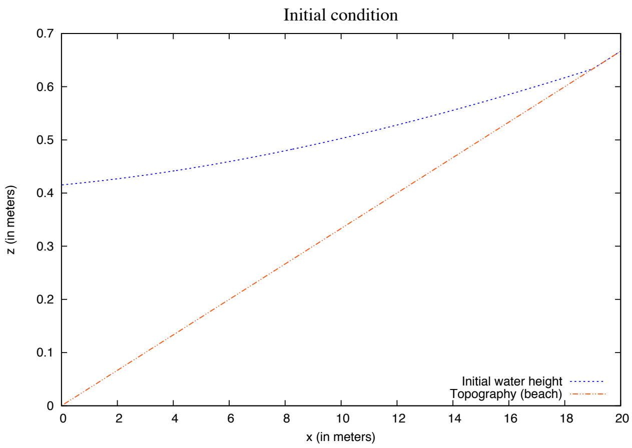

Figure 4: Initial condition (water height).

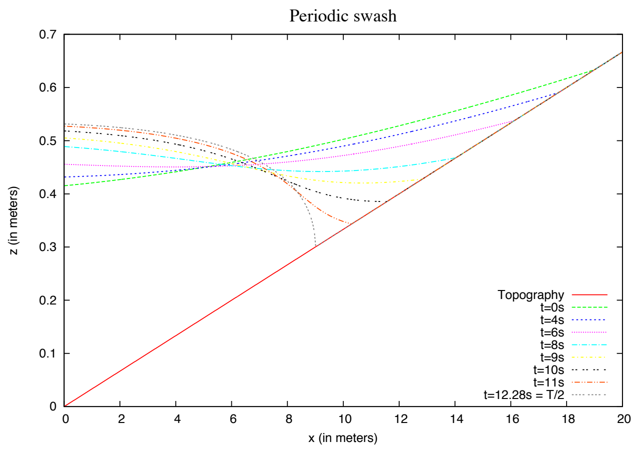

Figure 5: Water surface on half a period.

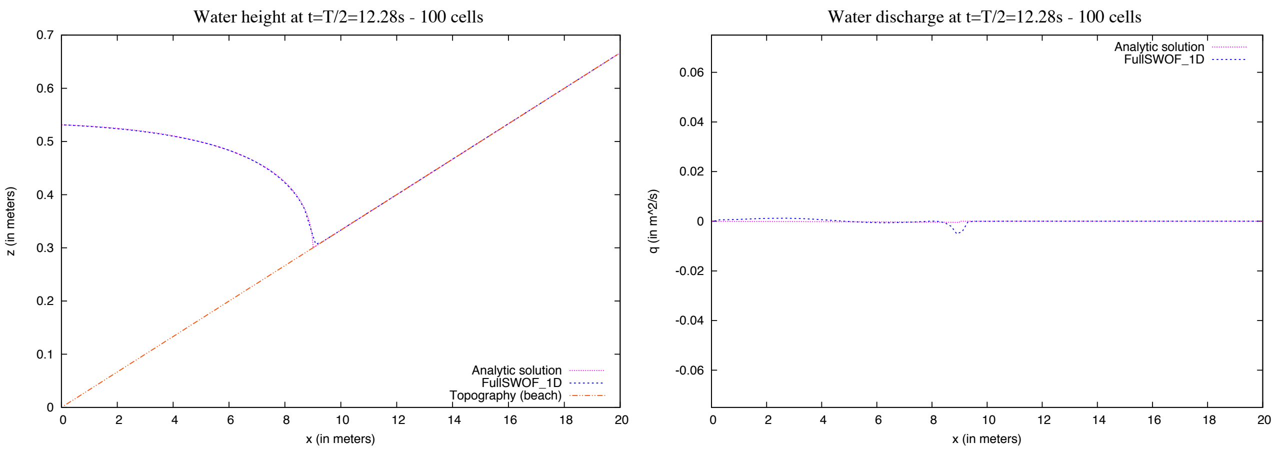

Figure 6: Results at $t_f = \frac{T}{2} = 12.28$ s.

The initial water height is the analytic solution at time $t_f = 0$ s, namely (see Figure 4):

$$\begin{cases} \eta(\sigma,0) = \dfrac{A}{4}\mathcal{J}_0(\sigma), \\ \dfrac{x}{L} = -\dfrac{\sigma^2}{16} + \eta(\sigma,0) + x_0, \\ h = \left(\eta + x_0 - \dfrac{x}{L}\right)\alpha L. \end{cases}$$

The flow moves with a periodic motion, see Figure 5, imposed by the periodic left boundary condition.

In *SWASHES*, $L = 20$ m, the slope of the beach $\alpha = \frac{1}{30}$, $x_0 = 0.7$ m and $e = 0.1$, see Figure 4.

The final time is the half period, $t_f = 12.28$ s and $A = 1$. On Figure 6, we compared the analytic solution with the results of *FullSWOF_1D* with 100 points: the water height is very well computed, the only difference is at the wet/dry transition and lowers when we refine the mesh. For the discharge, the difference seems, at first glance, more significant. However, the scale is chosen with the left imposed discharge value, that varies from -0.75 to 0.75 m^2/s. At the time $t_f = 12.28$ s, the flow stops and starts moving backwards: this transition is not easy to capture and it explains the differences. The refinement of the mesh also improves the results.

Remark 2 (Imposed height on the left boundary)**.** *The boundary condition "Imposed height" has also been modified locally in the code* FullSWOF_1D *to fit this example: it is defined from the file periodic_leftbc.txt given by* SWASHES *with nxcell=10000, namely:*

```
if (time<0.01){ hfix = 0.415214; qfix =-0.000203708; }
else if (time<0.02){ hfix = 0.415215; qfix =-0.000407415; }
else if (time<0.03){ hfix = 0.415215; qfix =-0.00061112; }
...
else if (time<12.26){ hfix = 0.531442; qfix =-0.000273717; }
else if (time<12.27){ hfix = 0.531443; qfix =-0.000162446; }
else { hfix = 0.531444; qfix =-5.11768e-05; }
```

References

[1] G. F. Carrier and H. P. Greenspan. Water waves of finite amplitude on a sloping beach. *Journal of Fluid Mechanics*, 4: 97–109, 1958. doi:10.1017/S0022112058000331. URL `http://journals.cambridge.org/article_S0022112058000331`.

[2] Noémie Gaveau. Étude et programmation de la solution analytique du *SWASH*. Stage 1A, É.N.S. de Rennes, 2015.